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Perturbation of Orthogonal Polynomials on an Arc of the Unit Circle, II*

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Orthogonal polynomials on the unit circle are fully determined by their reflection coefficients through the Szegő recurrences. Assuming that the reflection coefficients converge to a complex number a with $0 < |a| < 1$, or, in addition, they form a sequence of bounded variation, we analyze the orthogonal polynomials by comparing them with orthogonal polynomials with constant reflection coefficients which were studied earlier by Ya. L. Geronimus and N. I. Akhiezer. In particular, we present asymptotic relations under certain assumptions on the rate of convergence of the

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reflection coefficients. Under weaker conditions we still obtain useful information about the orthogonal polynomials and also about the measure of orthogonality.

Key words and phrases: Szegő orthogonal polynomials; unit circle orthogonal polynomials; reflection coefficients, perturbations.

1. INTRODUCTION

The present paper is a continuation of our study of polynomials orthogonal on an arc of the unit circle, started in [12]. We adopt here the notation used therein.[1]

Orthogonal polynomials $\{\varphi_n\}_0^\infty$ on the unit circle $\mathbb{T} \overset{\text{def}}{=} \{z \in \mathbb{C} : |z| = 1\}$ are defined by

$$\int_{\mathbb{T}} \varphi_n(\mu, z)\, \overline{\varphi_m(\mu, z)}\, d\mu(\vartheta) = \delta_{m,n}, \qquad z = e^{i\vartheta}, \quad m, n \in \mathbb{Z}^+,$$

where $\varphi_n(\mu, z) = \kappa_n(\mu)\, z^n + \text{lower degree terms}$ with $\kappa_n(\mu) > 0$ and μ is a probability measure in $[0, 2\pi)$ with infinite support. Here and in what follows we say that μ is a measure on $\mathbb{T}$, and, for a function f on $\mathbb{T}$, we set $\int_{\mathbb{T}} f\, d\mu \overset{\text{def}}{=} \int_0^{2\pi} f(e^{i\vartheta})\, d\mu(\vartheta)$. The monic orthogonal polynomials $\Phi_n \overset{\text{def}}{=} \kappa_n^{-1} \varphi_n$ along with the monic second kind orthogonal polynomials $\{\Psi_n \overset{\text{def}}{=} \kappa_n^{-1} \psi_n\}_0^\infty$ satisfy the (Szegő) recurrence relations

$$\begin{pmatrix} \Phi_{n+1} & \Psi_{n+1} \\ \Phi_{n+1}^* & -\Psi_{n+1}^* \end{pmatrix} = \begin{pmatrix} z & a_{n+1} \\ z\bar{a}_{n+1} & 1 \end{pmatrix} \begin{pmatrix} \Phi_n & \Psi_n \\ \Phi_n^* & -\Psi_n^* \end{pmatrix}, \qquad n \in \mathbb{Z}^+, \tag{1}$$

where $\Phi_0 \equiv 1$, $\Psi_0 \equiv 1$, $a_n \overset{\text{def}}{=} \Phi_n(0)$ (cf. [12, formula (8)]), and the reversed $*$-polynomial of a polynomial ρ_n of degree n is defined by $\rho_n^*(z) \overset{\text{def}}{=} z^n \bar{\rho}_n(z^{-1})$. Note that the monic second kind orthogonal polynomials $\{\Psi_n\}_0^\infty$ are determined by replacing a_n with $-a_n$ in the recurrences for $\{\Phi_n\}_0^\infty$ and $\{\Phi_n^*\}_0^\infty$. The elements of the sequence $\{a_n\}_0^\infty$ are called reflection coefficients and/or Szegő and/or Schur parameters. We can relate the leading coefficients $\{\kappa_n\}_0^\infty$ to the reflection coefficients $\{a_n\}_0^\infty$ via

$$\sum_{k=0}^{n} |\varphi_k(0)|^2 = \kappa_n^2, \qquad n \in \mathbb{Z}^+,$$

[1] In what follows, whenever it does not lead to confusion, we will suppress arguments such as z (as in $\varphi_n(\mu, z)$) to simplify the notation. We write $\mathbb{Z}^+ \overset{\text{def}}{=} \{n \in \mathbb{Z} : n \geqslant 0\}$ and $\mathbb{R}^+ \overset{\text{def}}{=} \{x \in \mathbb{R} : x \geqslant 0\}$.

and

$$\frac{\kappa_n^2}{\kappa_{n+1}^2} = 1 - |a_{n+1}|^2, \qquad n \in \mathbb{Z}^+, \tag{2}$$

(cf. [8, formula (1.5), p. 7 or formula (1.9), p. 9]). By the analogue of Favard's theorem on $\mathbb{T}$ (cf. [4]), an arbitrary sequence $\{a_n\}_0^\infty$ with $a_0 \stackrel{\text{def}}{=} 1$ and $|a_n| < 1$ for $n \in \mathbb{N}$, completely determines the sequence of orthogonal polynomials $\{\Phi_n\}_0^\infty$. In fact, given such a sequence $\{a_n\}_0^\infty$, the polynomials $\{\Phi_n(\mu)\}_0^\infty$ obtained by the Szegő recurrences are orthogonal with respect to a unique probability measure μ on $\mathbb{T}$ with infinite support, such that $\Phi_n(\mu, 0) = a_n$ for $n \in \mathbb{Z}^+$. A very special case is the sequence of Geronimus polynomials $\{\hat{\Phi}_n\}_0^\infty$, where $a_n \stackrel{\text{def}}{=} a$ for $n \in \mathbb{N}$ with $0 < |a| < 1$. Analogously, we can talk about the sequences $\{\hat{\varphi}_n\}_0^\infty$, $\{\hat{\psi}_n\}_0^\infty$, and $\{\hat{\Psi}_n\}_0^\infty$ as well.

We view the Geronimus polynomials as the unperturbed polynomials, while $\{\varphi_n\}_0^\infty$ corresponding to $\{a_n\}_0^\infty$ are the perturbed ones. Our goal is to describe the perturbed system of orthogonal polynomials in comparison to the unperturbed system when some restraints are placed on the convergence behavior of $\{a_n\}_0^\infty$.[2]

The Geronimus polynomials essentially live on an arc of the unit circle characterized by α, such that

$$\sin(\alpha/2) \stackrel{\text{def}}{=} |a|, \qquad \alpha \in (0, \pi), \tag{3}$$

(cf. [12, Sect. 2]). For $\beta \in (0, \pi)$ we define

$$\begin{aligned} \Delta_\beta &\stackrel{\text{def}}{=} \{e^{i\vartheta} : \beta \leqslant \vartheta \leqslant 2\pi - \beta\}, \\ \Delta_\beta^o &\stackrel{\text{def}}{=} \{e^{i\vartheta} : \beta < \vartheta < 2\pi - \beta\}, \\ \Delta_\beta^c &\stackrel{\text{def}}{=} \{e^{i\vartheta} : -\beta < \vartheta < \beta\}. \end{aligned} \tag{4}$$

Using this terminology, the support of the orthogonality measure $\hat{\mu}$ corresponding to $\{\hat{\varphi}_n\}_0^\infty$ consists of Δ_α and one possible mass point in $\Delta_\alpha{}^c$.

In [12] the matrix recurrences (1) were used to manage the computations. However, the matrix recurrences were not ideal to handle certain improvements of [12, Theorem 12, p. 410], such as asymptotics for $\{\varphi_n\}_0^\infty$ at $z = e^{\pm i\alpha}$ under the condition $\sum_{n=0}^\infty n\,|a_n - a| < \infty$.[3] On the other hand, if the matrix approach works, it may still be possible to replace it with an argument involving three-term recurrences. For example, [12, Theorem 12, p. 410] may be proved by combining the technique of reducing the order of the three-term recurrences used in [19, 20] with a trigonometric Schur-type inequality (cf. [5, Theorem 6, p. 85]).

[2] The same problem in a different but more general context is treated in [23].

[3] For the continuous analogue of this condition in the spectral theory of the Schrödinger operator see, for instance, [1, Chap. II, formula (2.1.2), p. 37].

The three-term recurrence relation for $\{\Phi_n\}_0^\infty$ can easily be deduced from (1) (cf. [9, formula (3.4), p. 4]). We write it as

$$a_{n+1}\, y_{n+2}(z) - (a_{n+1} z + a_{n+2})\, y_{n+1}(z) + z a_{n+2}(1 - |a_{n+1}|^2)\, y_n(z) = 0, \quad (5)$$

where $y_n(z) \stackrel{\text{def}}{=} \Phi_n(z)$, with initial conditions $\Phi_0 \equiv 1$ and $\Phi_1(z) = z + a_1$. It is easy to see that $y_n(z) \stackrel{\text{def}}{=} \Psi_n(z)$ also satisfies (5) with $\Psi_0 \equiv 1$ and $\Psi_1(z) = z - a_1$. Let $N_0 \in \mathbb{N}$ be defined by

$$N_0 \stackrel{\text{def}}{=} \min\{k \in \mathbb{Z}^+ : a_{n+1} \neq 0 \text{ for every } n \geqslant k\}. \quad (6)$$

In this paper the index N_0 will exist since $\lim_{n\to\infty} a_n = a \neq 0$ is always going to be assumed. Now consider

$$y_{n+2}(z) - \left(z + \frac{a_{n+2}}{a_{n+1}}\right) \frac{\kappa_{n+2}}{\kappa_{n+1}}\, y_{n+1}(z) + z\, \frac{a_{n+2}}{a_{n+1}} \frac{\kappa_n \kappa_{n+2}}{\kappa_{n+1}^2}\, y_n(z) = 0, \qquad n \geqslant N_0, \quad (7)$$

which, by (2), is equivalent to (5) when $n \geqslant N_0$. Then $\{\varphi_n\}_{N_0}^\infty$ and $\{\psi_n\}_{N_0}^\infty$ form a fundamental set of solutions to (7) for the range $n \geqslant N_0$. This follows from the expression

$$\begin{vmatrix} \varphi_n(z) & \psi_n(z) \\ \varphi_{n+1}(z) & \psi_{n+1}(z) \end{vmatrix} = -\frac{2 a_{n+1} \kappa_{n+1}}{\kappa_n}\, z^n, \qquad n \in \mathbb{Z}^+, \quad (8)$$

for the Wronskian which doesn't vanish for $n \geqslant N_0$. We also mention that $\{\Phi_n\}_0^\infty$ and $\{\Psi_n\}_0^\infty$ form a fundamental set of solutions for

$$\bar{a}_{n+1}\, y_{n+2}(z) - (\bar{a}_{n+1} + \bar{a}_{n+2}\, z)\, y_{n+1}(z) + \bar{a}_{n+2} z(1 - |a_{n+1}|^2)\, y_n(z) = 0$$

(cf. [8, formula (8.9), p. 157]) where analogous special attention needs to be paid to the case when $a_{n+1} = 0$.

The following is a well known fact about the general solution of second order linear difference equations (see [6, Sect. 5.3.5, formula (30), p. 308 (Russian), p. 305 (French), p. 368 (English)]).

PROPOSITION 1. *Assume that* $\{f_1(n)\}_0^\infty$ *and* $\{f_2(n)\}_0^\infty$ *satisfy the homogeneous difference equation*

$$P_0(n)\, f(n+2) + P_1(n)\, f(n+1) + P_2(n)\, f(n) = 0, \qquad n \in \mathbb{Z}^+,$$

and there is $n_0 \in \mathbb{Z}^+$ *such that*

$$\begin{vmatrix} f_1(n_0) & f_2(n_0) \\ f_1(n_0+1) & f_2(n_0+1) \end{vmatrix} \neq 0.$$

Then, assuming $P_0(n) \neq 0$ for $n \geqslant n_0$, the general solution of

$$P_0(n)\, f(n+2) + P_1(n)\, f(n+1) + P_2(n)\, f(n) = Q(n), \qquad n \geqslant n_0,$$

can be expressed in the form

$$f(n) = \sum_{k=n_0}^{n-1} \frac{\begin{vmatrix} f_1(k+1) & f_2(k+1) \\ f_1(n) & f_2(n) \end{vmatrix}}{\begin{vmatrix} f_1(k+1) & f_2(k+1) \\ f_1(k+2) & f_2(k+2) \end{vmatrix}} \frac{Q(k)}{P_0(k)} + c_1 f_1(n) + c_2 f_2(n), \qquad n \geqslant n_0,$$

where c_1 and c_2 are arbitrary constants. Given initial conditions $f(n_0)$ and $f(n_0+1)$, the constants c_1 and c_2 can be determined from

$$f(j) = c_1 f_1(j) + c_2 f_2(j), \qquad j = n_0, \quad n_0 + 1.$$

We will also need Gronwall's inequality (cf. [17, Lemma 3.2, p. 21; 14, Lemma 4, p. 250; 28, p. 440]).

PROPOSITION 2. *Given $\sigma_1 \in \mathbb{Z}$ and $\sigma_2 \in \mathbb{Z}$ with $\sigma_1 < \sigma_2$, if the sequences $\{u_n \geqslant 0\}_{n=\sigma_1}^{\sigma_2}$ and $\{v_n \geqslant 0\}_{n=\sigma_1}^{\sigma_2}$ satisfy*

$$u_n \leqslant d + \sum_{k=\sigma_1}^{n-1} v_k\, u_k, \qquad \sigma_1 \leqslant n \leqslant \sigma_2,$$

then

$$u_n \leqslant d \exp\left(\sum_{k=\sigma_1}^{n-1} v_k \right), \qquad \sigma_1 \leqslant n \leqslant \sigma_2.$$

COROLLARY 3. *Given $\sigma_1 \in \mathbb{Z}$ and $\sigma_2 \in \mathbb{Z}$ with $\sigma_1 < \sigma_2$, if the sequences $\{u_n \geqslant 0\}_{n=\sigma_1}^{\sigma_2}$ and $\{0 \leqslant v_n < 1\}_{n=\sigma_1}^{\sigma_2}$ satisfy*

$$u_n \leqslant d + \sum_{k=\sigma_1}^{n} v_k\, u_k, \qquad \sigma_1 \leqslant n \leqslant \sigma_2, \tag{9}$$

then

$$u_n \leqslant \frac{d}{1 - v_n} \exp\left(\sum_{k=\sigma_1}^{n-1} \frac{v_k}{1 - v_k} \right), \qquad \sigma_1 \leqslant n \leqslant \sigma_2.$$

Proof. Rewrite (9) as

$$(1-v_n)\, u_n \leqslant d + \sum_{k=\sigma_1}^{n-1} \frac{v_k}{1-v_k}\,(1-v_k)\, u_k, \qquad \sigma_1 \leqslant n \leqslant \sigma_2,$$

and apply Proposition 2. ▮

The next result establishes the asymptotic behavior of the solutions of certain second order difference inequalities with two identical characteristic roots. The proof of this and its higher order analogues can be found in [14, Theorem 3, p. 247].

PROPOSITION 4. *Given* $n_0 \in \mathbb{Z}$, *let* $f\colon \mathbb{Z} \to \mathbb{C}$ *vanish in* $(-\infty, n_0) \cap \mathbb{Z}$. *Suppose that* f *satisfies the difference inequality*

$$|f(n+2) - 2f(n+1) + f(n)| \leqslant g(n)(|f(n)| + |f(n+1)| + |f(n+2)|)$$

for every integer $n \geqslant n_0$ *with* $\{g\colon \mathbb{Z} \to \mathbb{R}^+\}$ *satisfying*

$$\sum_{k=n_0}^{\infty} g(k)\, k < \infty.$$

Then either $f(n) = 0$ *starting with a sufficiently large index* n *or else, either for* $r = 0$ *or for* $r = 1$, $\lim_{n\to\infty} n^{-r} f(n)$ *exists and it is different from* 0.

2. THE CASE OF CONSTANT REFLECTION COEFFICIENTS

In this section we present explicit formulas for the Geronimus polynomials $\{\hat{\varphi}_n\}_0^\infty$ (cf. [12, Sect. 2]) which will help us how to establish asymptotic results for the perturbed polynomials. We assume that $0 < |a| < 1$ and that α is determined by (3). Let z_1 and z_2 denote the zeros of

$$w^2 - (z+1)\, w + (1-|a|^2)\, z = 0. \tag{10}$$

Then

$$z_1 = \frac{z+1+\sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})}}{2} \quad \text{and} \quad z_2 = \frac{z+1-\sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})}}{2}, \tag{11}$$

where the branch of the square root is chosen such that

$$\lim_{z\to\infty} \frac{\sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})}}{z} = 1.$$

We will frequently use the notation

$$r_{1,2} \stackrel{\text{def}}{=} z_{1,2}/\sqrt{1-|a|^2}. \tag{12}$$

Using (1) we can write

$$\begin{aligned}\begin{pmatrix}\hat{\Phi}_{n+1} & \hat{\Psi}_{n+1}\\ \hat{\Phi}^*_{n+1} & \hat{\Psi}^*_{n+1}\end{pmatrix} &= \begin{pmatrix} z & a\\ z\bar{a} & 1\end{pmatrix}\begin{pmatrix}\hat{\Phi}_n & \hat{\Psi}_n\\ \hat{\Phi}^*_n & \hat{\Psi}^*_n\end{pmatrix}\\ &= \begin{pmatrix} z & a\\ z\bar{a} & 1\end{pmatrix}^{n+1}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}, \qquad n\in\mathbb{Z}^+,\end{aligned} \tag{13}$$

that is,

$$\hat{\varphi}_{n+2} - \frac{z+1}{\sqrt{1-|a|^2}}\,\hat{\varphi}_{n+1} + z\hat{\varphi}_n = 0, \qquad n\in\mathbb{Z}^+, \tag{14}$$

(cf. (7)).

Case 1. $z \stackrel{\text{def}}{=} e^{i\vartheta} \in \Delta^o_\alpha$. Then $|r_1| = |r_2|$ but $r_1 \neq r_2$.
By (11)

$$z_{1,2} = e^{i(\vartheta/2)}\left(\cos\frac{\vartheta}{2} \pm i\sqrt{\sin\frac{\vartheta-\alpha}{2}\sin\frac{\vartheta+\alpha}{2}}\right).$$

In particular, $|z_1| = |z_2| = (1-|a|^2)^{1/2}$ and $|r_1| = |r_2| = 1$. We can use the characteristic equation (10) to evaluate the matrix power in (13), which yields (cf. [12, Sect. 2, p. 399])

$$\hat{\varphi}_n = \frac{\hat{A}z_1^n + \hat{B}z_2^n}{(1-|a|^2)^{n/2}} = \hat{A}r_1^n + \hat{B}r_2^n \qquad \text{and} \qquad \hat{\varphi}^*_n = \frac{\hat{C}z_1^n + \hat{D}z_2^n}{(1-|a|^2)^{n/2}} = \hat{C}r_1^n + \hat{D}r_2^n, \tag{15}$$

where $\hat{A}$, $\hat{B}$, $\hat{C}$, and $\hat{D}$ are functions of z which do not depend on n. For $\hat{\psi}_n$ and $\hat{\psi}^*_n$ similar representations hold as well.

Case 2. $z \stackrel{\text{def}}{=} e^{i\vartheta} = e^{\pm i\alpha}$. Then $r_1 = r_2$.

We restrict ourselves to the case when $\vartheta = \alpha$. If $\vartheta = -\alpha$, a single sign change will suffice to obtain the corresponding results. By (11)

$$z_{1,2} = \frac{z+1}{2} = \frac{e^{i\alpha}+1}{2} = (1-|a|^2)^{1/2}\, e^{i(\alpha/2)},$$

so that $|r_1|=|r_2|=1$, and from (15) (see also [12, formulas (17) and (18), p. 399]), after the limit $\vartheta \to \alpha+0$ is taken,

$$\hat{\varphi}_n(e^{i\alpha}) = e^{i(\alpha/2)n}\left[\left(\frac{2(e^{i\alpha}+a)}{e^{i\alpha}+1}-1\right)n+1\right]$$

and

$$\hat{\varphi}_n^*(e^{i\alpha}) = e^{i(\alpha/2)n}\left[\left(\frac{2(1+\bar{a}e^{i\alpha})}{e^{i\alpha}+1}-1\right)n+1\right].$$

There are analogous formulas for $\hat{\psi}_n(e^{i\alpha})$ and $\hat{\psi}_n^*(e^{i\alpha})$.

Case 3. $z \overset{\text{def}}{=} e^{i\vartheta} \in \Delta_\alpha^c$. Then $|r_1|>|r_2|$.

In this case (15) remains valid. As for the absolute values of r_1 and r_2, by (11), we have

$$z_{1,2} = e^{i(\vartheta/2)}\left(\cos\frac{\vartheta}{2} \pm \sqrt{\sin\frac{\alpha-\vartheta}{2}\sin\frac{\alpha+\vartheta}{2}}\right),$$

so that

$$\begin{aligned}|z_{1,2}| &= \cos\frac{\vartheta}{2} + \sqrt{\sin\frac{\alpha-\vartheta}{2}\sin\frac{\alpha+\vartheta}{2}}\\ &= \sqrt{\frac{1+\cos\vartheta}{2}} \pm \sqrt{\frac{\cos\vartheta-\cos\alpha}{2}}\\ &= \frac{1-|a|^2}{\sqrt{(1+\cos\vartheta)/2} \mp \sqrt{(\cos\vartheta-\cos\alpha)/2}},\end{aligned}$$

from which

$$\frac{\sqrt{1-|a|^2}}{2} \leqslant \frac{1-|a|}{\sqrt{1-|a|^2}} \leqslant |r_2| < 1,$$

and

$$1 < |r_1| \leqslant \frac{2}{\sqrt{1-|a|^2}}.$$

We point out two more useful facts about the Geronimus polynomials in the special case when $|a-1/2|=1/2$. The first one is the explicit formula

$$\hat{\varphi}_n(e^{i\vartheta}) = e^{i\left(\frac{\vartheta}{2}\right)(n-1)}\left(\frac{\sin(n+1)\lambda}{\sin\lambda}e^{i\left(\frac{\vartheta}{2}\right)} - \frac{\sin n\lambda}{\sin\lambda}e^{i\left(\frac{\alpha}{2}\right)}\right), \qquad n\in\mathbb{Z}^+, \tag{16}$$

(cf. [8, formula (4.14′), p. 50]) where the parameter $\lambda \in [0, \pi]$ is given by

$$\cos \lambda \overset{\text{def}}{=} \frac{\cos(\vartheta/2)}{\cos(\alpha/2)}, \qquad \alpha \leqslant \vartheta \leqslant 2\pi - \alpha.$$

The second one is about the asymptotic behavior of the Christoffel function $\hat{K}_m(z, z) \overset{\text{def}}{=} \sum_{j=0}^{m} |\hat{\varphi}_j(z)|^2$ (cf. (52)). By (16), for $|a - 1/2| = 1/2$ and $z \overset{\text{def}}{=} e^{i\vartheta} \in \Delta_\alpha^o$, we have

$$\sin^2 \lambda \hat{K}_n(z, z)$$
$$= \sin^2(n+1)\lambda + 2 \sum_{j=1}^{n} \sin^2 j\lambda - 2 \cos \frac{\vartheta - \alpha}{2} \sum_{j=1}^{n} \sin(j+1)\lambda \sin j\lambda.$$

Since

$$\sum_{j=1}^{n} \sin^2 j\lambda = \frac{n}{2} - \frac{1}{2} \sum_{j=1}^{n} \cos 2j\lambda$$

and

$$\sum_{j=1}^{n} \sin(j+1)\lambda \sin j\lambda = \frac{\cos \lambda}{2} \left(n - \sum_{j=1}^{n} \cos 2j\lambda \right) + \frac{\sin \lambda}{2} \sum_{j=1}^{n} \sin 2j\lambda,$$

and

$$\sum_{j=1}^{n} \cos jx = \frac{\sin((n+1)/2)\, x \cos(n/2)\, x}{\sin(x/2)} - 1$$

and

$$\sum_{j=1}^{n} \sin jx = \frac{\sin((n+1)/2)\, x \sin(n/2)\, x}{\sin(x/2)},$$

we have

$$\lim_{n \to \infty} \frac{\hat{K}_n(z, z)}{n} = \frac{1 - \cos((\vartheta - \alpha)/2) \cos \lambda}{\sin^2 \lambda} = |1 - a| \frac{\sin(\vartheta/2)}{\sin((\vartheta + \alpha)/2)}, \tag{17}$$

where the convergence is locally uniform in Δ_α^o.[4]

[4] N.B. that the latter limit relation appeared for the first time in [10, formula (4.13), p. 49], where it was derived from [10, Theorem 3.2, p. 46] whose proof (very unfortunately) contains an error (cf. [21, Section 4.6, pp. 26–28]).

3. PERTURBATION OF THE ROOTS OF THE CHARACTERISTIC EQUATION

Recall that $z=e^{i\vartheta}$, $|a_n|<1$ for $n\in\mathbb{N}$, $0<|a|<1$, α is defined by (3), $\lim_{n\to\infty} a_n=a$, and N_0 is defined in (6). It will be convenient and later helpful if we rewrite (7) for the orthonormal polynomials as

$$\varphi_{n+2}-(r_{1,n}+r_{2,n})\,\varphi_{n+1}+r_{1,n}r_{2,n}\,\varphi_n=0, \qquad n\geqslant N_0, \tag{18}$$

and, similarly, for the orthonormal Geronimus polynomials (cf. (14)),

$$\hat{\varphi}_{n+2}-(r_1+r_2)\,\hat{\varphi}_{n+1}+r_1r_2\,\hat{\varphi}_n=0, \qquad n\geqslant 0, \tag{19}$$

where we denote by $r_{1,n}$ and $r_{2,n}$ the roots of the characteristic equation

$$r^2-\left(z+\frac{a_{n+2}}{a_{n+1}}\right)\frac{\kappa_{n+2}}{\kappa_{n+1}}\,r+z\,\frac{a_{n+2}}{a_{n+1}}\,\frac{\kappa_n\kappa_{n+2}}{\kappa_{n+1}^2}=0, \qquad n\geqslant N_0, \tag{20}$$

of the linear recurrence (7), so that

$$r_{1,n}+r_{2,n}=\left(z+\frac{a_{n+2}}{a_{n+1}}\right)\frac{\kappa_{n+2}}{\kappa_{n+1}} \qquad \text{and} \qquad r_{1,n}r_{2,n}=z\,\frac{a_{n+2}}{a_{n+1}}\,\frac{\kappa_n\kappa_{n+2}}{\kappa_{n+1}^2}. \tag{21}$$

In particular, for $z\in\mathbb{T}$

$$r_{1,n}\neq 0 \qquad \text{and} \qquad r_{2,n}\neq 0, \qquad n\geqslant N_0,$$

and, if $0<\inf_{n>N_0}|a_n|\leqslant\sup_{n\geqslant N_0}|a_n|<1$, then there is a constant K_0 such that

$$\sup_{z\in\mathbb{T}}\ \sup_{N_0\leqslant m_1\leqslant m_2}\ \prod_{n=m_1}^{m_2}|r_{1,n}r_{2,n}|^{\pm 1}<K_0. \tag{22}$$

For the orthonormal Geronimus polynomials, (21) reduces to

$$r_1+r_2=(z+1)(1-|a|^2)^{-1/2} \qquad \text{and} \qquad r_1r_2=z. \tag{23}$$

Our results are based on the convergence behavior of $\{a_n\}_0^\infty$. We will formulate these conditions in terms of the roots of the characteristic polynomials of (18) and (19). This is accomplished in two steps. First we express these conditions in terms of the coefficients of (18) and (19) (see (24) and (25)), and then we move from the coefficients of (18) and (19) to the roots of the corresponding characteristic polynomials. This approach works as long as the roots are different (see (27) and (28)). We omit most of the details for they are tedious but simple.

For $z \in \Delta_\alpha$ (in fact, uniformly in the convex hull of Δ_α), there exist functions E_1, E_2, and E_3 depending on a, and for every fixed $\varepsilon > 0$ there is $N_1(\varepsilon) \geqslant N_0$ such that

$$\begin{aligned} |r_{1,n} + r_{2,n} - r_1 - r_2| &\leqslant (E_1 + \varepsilon)\,|a_{n+1} - a| + (E_2 + \varepsilon)\,|a_{n+2} - a|, \\ |r_{1,n}\, r_{2,n} - r_1\, r_2| &\leqslant (E_3 + \varepsilon)(|a_{n+1} - a| + |a_{n+2} - a|), \qquad n \geqslant N_1(\varepsilon). \end{aligned} \tag{24}$$

In Section 6, we will need a similar pair of inequalities for the roots of the characteristic polynomial (20) written as

$$\begin{aligned} &|r_{1,n} + r_{2,n} - r_{1,n+1} - r_{2,n+1}| \\ &\qquad \leqslant (E_1 + \varepsilon)\,|a_{n+2} - a_{n+1}| + (E_2 + \varepsilon)\,|a_{n+3} - a_{n+2}|, \\ &|r_{1,n}\, r_{2,n} - r_{1,n+1}\, r_{2,n+1}| \\ &\qquad \leqslant (E_3 + \varepsilon)(|a_{n+2} - a_{n+1}| + |a_{n+3} - a_{n+2}|), \qquad n \geqslant N_1(\varepsilon). \end{aligned} \tag{25}$$

For instance,

$$\begin{aligned} E_1(a) &\stackrel{\text{def}}{=} (1 - |a|^2)^{-1/2}\,|a|^{-1}, \\ E_2(a) &\stackrel{\text{def}}{=} (1 - |a|^2)^{-1/2}\,|a|^{-1} + 2\,|a|\,(1 - |a|^2)^{-1}, \\ E_3(a) &\stackrel{\text{def}}{=} |a|^{-1} + |a|\,(1 - |a|^2)^{-1} \end{aligned} \tag{26}$$

are appropriate choices for (24) and (25) to hold. To see how one arrives at such estimates, we will derive the first inequality in (24). From (21) and (23) we find

$$\begin{aligned} |r_{1,n} + r_{2,n} - r_1 - r_2| &= \left| \left(z + \frac{a_{n+2}}{a_{n+1}} \right) \frac{\kappa_{n+2}}{\kappa_{n+1}} - (z+1) \frac{1}{\sqrt{1 - |a|^2}} \right| \\ &= \left| (z+1) \left(\frac{\kappa_{n+2}}{\kappa_{n+1}} - \frac{1}{\sqrt{1 - |a|^2}} \right) + \left(\frac{a_{n+2}}{a_{n+1}} - 1 \right) \frac{\kappa_{n+2}}{\kappa_{n+1}} \right| \\ &\leqslant 2 \sqrt{1 - |a|^2} \left| \frac{1}{\sqrt{1 - |a_{n+2}|^2}} - \frac{1}{\sqrt{1 - |a|^2}} \right| \\ &\quad + \frac{1}{\sqrt{1 - |a_{n+2}|^2}} \left| \frac{a_{n+2}}{a_{n+1}} - 1 \right|, \end{aligned}$$

where we have used that $|z+1| \leqslant 2\sqrt{1 - |a|^2}$ in the convex hull of Δ_α. Now use

$$\left| \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{y}} \right| = \frac{|x - y|}{(\sqrt{x} + \sqrt{y})\,\sqrt{xy}}$$

to obtain

$$\left|\frac{1}{\sqrt{1-|a_{n+2}|^2}}-\frac{1}{\sqrt{1-|a|^2}}\right| \leqslant \frac{||a_{n+2}|^2-|a|^2|}{2(\sqrt{1-|a|^2})^3+o(1)}$$
$$\leqslant \frac{|a_{n+2}-a|\,(|a_{n+2}|+|a|)}{2(\sqrt{1-|a|^2})^3+o(1)}$$
$$\leqslant \left(\frac{|a|}{(\sqrt{1-|a|^2})^3}+o(1)\right)|a_{n+2}-a|.$$

Furthermore,

$$\left|\frac{a_{n+2}}{a_{n+1}}-1\right| = \frac{1}{|a_{n+1}|}\,|a_{n+2}-a_{n+1}| \leqslant \frac{1}{|a_{n+1}|}\,(|a_{n+1}-a|+|a_{n+2}-a|).$$

Hence,

$$|r_{1,n}+r_{2,n}-r_1-r_2|$$
$$\leqslant \left(\frac{2\,|a|}{1-|a|^2}+o(1)\right)|a_{n+2}-a|$$
$$+\left(\frac{1}{|a|\,\sqrt{1-|a|^2}}+o(1)\right)(|a_{n+1}-a|+|a_{n+2}-a|),$$

giving the first inequality in (24). The other inequalities in (24) and (25) follow by similar estimates.

In order to move on from the coefficients to the roots, we use

$$r_{j,n}=\frac{r_{1,n}+r_{2,n}+(-1)^{j+1}\sqrt{(r_{1,n}+r_{2,n})^2-4r_{1,n}\,r_{2,n}}}{2},$$
$$r_j=\frac{r_1+r_2+(-1)^{j+1}\sqrt{(r_1+r_2)^2-4r_1\,r_2}}{2}, \qquad j=1,2.$$

The following inequalities are not uniformly valid on Δ_α since they break down at the endpoints $e^{\pm i\alpha}$. Thus, in practice, we use them on a compact subsets of Δ_α^o. Given $\Delta=\bar{\Delta}\subset\Delta_\alpha^o$ and $\varepsilon>0$, there is $N_2(\varepsilon,\Delta)\geqslant N_1(\varepsilon)$ such that for $j=1$ and $j=2$ the inequalities

$$|r_{j,n}-r_j| \leqslant |r_1-r_2|^{-1}\,\{(E_4+\varepsilon)\,|a_{n+1}-a|+(E_5+\varepsilon)\,|a_{n+2}-a|\}, \tag{27}$$

and (cf. (25))

$$|r_{j,n}-r_{j,n+1}| \leqslant |r_1-r_2|^{-1}\,\{(E_4+\varepsilon)\,|a_{n+2}-a_{n+1}|+(E_5+\varepsilon)\,|a_{n+3}-a_{n+2}|\} \tag{28}$$

hold for $n \geqslant N_2(\varepsilon, \Delta)$. The expressions

$$E_4 \stackrel{\text{def}}{=} 2E_1 + E_3 \qquad \text{and} \qquad E_5 \stackrel{\text{def}}{=} 2E_2 + E_3$$

are appropriate choices for the above defined functions.

4. ASYMPTOTIC ANALYSIS

A Solution Formula for the Perturbed Equation. First we establish a connection between (18) and (19) by rewriting (18) as a constant coefficient non-homogeneous equation

$$\varphi_{n+2} - (r_1 + r_2)\,\varphi_{n+1} + r_1 r_2\,\varphi_n = Q_n, \tag{29}$$

where

$$Q_n \stackrel{\text{def}}{=} (r_{1,n} + r_{2,n} - r_1 - r_2)\,\varphi_{n+1} - (r_{1,n} r_{2,n} - r_1 r_2)\,\varphi_n.$$

In what follows, given $\varepsilon > 0$, let $n_0 \stackrel{\text{def}}{=} N_1(\varepsilon)$ so that the inequalities in (24) hold. If $z \neq e^{\pm i\alpha}$, then r_1^n and r_2^n form a fundamental set of solutions to the homogeneous form of (29). Thus, by Proposition 1,

$$\begin{aligned}
\varphi_n &= \sum_{k=n_0}^{n-1} \frac{\begin{vmatrix} r_1^{k+1} & r_2^{k+1} \\ r_1^n & r_2^n \end{vmatrix}}{\begin{vmatrix} r_1^{k+1} & r_2^{k+1} \\ r_1^{k+2} & r_2^{k+2} \end{vmatrix}} Q_k + c_1 r_1^n + c_2 r_2^n \\
&= \sum_{k=n_0}^{n-2} \frac{r_2^{n-k-1} - r_1^{n-k-1}}{r_2 - r_1} [(r_{1,k} + r_{2,k} - r_1 - r_2)\varphi_{k+1} - (r_{1,k} r_{2,k} - r_1 r_2)\varphi_k] \\
&\quad + c_1 r_1^n + c_2 r_2^n \\
&= \sum_{k=n_0+1}^{n-1} \frac{r_2^{n-k} - r_1^{n-k}}{r_2 - r_1} (r_{1,k-1} + r_{2,k-1} - r_1 - r_2)\,\varphi_k \\
&\quad + \frac{r_2^{n-n_0} - r_1^{n-n_0}}{r_2 - r_1}\,\varphi_{n_0+1} \\
&\quad - \sum_{k=n_0}^{n-2} \frac{r_2^{n-k-1} - r_1^{n-k-1}}{r_2 - r_1} (r_{1,k} r_{2,k} - r_1 r_2)\,\varphi_k \\
&\quad - r_1 r_2 \frac{r_2^{n-n_0-1} - r_1^{n-n_0-1}}{r_2 - r_1}\,\varphi_{n_0},
\end{aligned} \tag{30}$$

where, on the right hand side, the limit value is to be taken if $z = e^{\pm i\alpha}$.

It is also essential to establish bounds for $\{\varphi_n\}_0^\infty$. To this end we start with

$$|\varphi_n(z)| \leqslant d_n(z) + \sum_{k=n_0}^{n-1} v_{kn}(z)\, |\varphi_k(z)|, \qquad |z|=1,\ n \geqslant n_0, \tag{31}$$

which is a consequence of (30). Here

$$d_n \stackrel{\text{def}}{=} \left| \frac{r_2^{n-n_0} - r_1^{n-n_0}}{r_2 - r_1} \right| |\varphi_{n_0+1}| + \left| r_1 r_2 \frac{r_2^{n-n_0-1} - r_1^{n-n_0-1}}{r_2 - r_1} \right| |\varphi_{n_0}| \tag{32}$$

and

$$\begin{aligned} v_{kn} \stackrel{\text{def}}{=} & \left| \frac{r_2^{n-k} - r_1^{n-k}}{r_2 - r_1} (r_{1,k-1} + r_{2,k-1} - r_1 - r_2) \right| \\ & + \left| \frac{r_2^{n-k-1} - r_1^{n-k-1}}{r_2 - r_1} (r_{1,k} r_{2,k} - r_1 r_2) \right|. \end{aligned} \tag{33}$$

We will also use

$$v_k \stackrel{\text{def}}{=} |r_{1,k-1} + r_{2,k-1} - r_1 - r_2| + |r_{1,k} r_{2,k} - r_1 r_2|. \tag{34}$$

Now we are ready to formulate the first main result of this section.

THEOREM 5. *Let* $|a_n| < 1$ *for* $n \in \mathbb{N}$, $0 < |a| < 1$, $\sin(\alpha/2) \stackrel{\text{def}}{=} |a|$ *with* $\alpha \in (0, \pi)$, *and let* $\{\varphi_n\}_0^\infty$ *be a solution of* (7) (*cf.* (18)).

(1) *If* $\sum_{n=0}^\infty |a_n - a| < \infty$ *and* $\Delta = \bar{\Delta} \subset \Delta_\alpha^o$, *then there exist two functions* $A_\infty \in C(\Delta_\alpha^o)$ *and* $B_\infty \in C(\Delta_\alpha^o)$ *such that*

$$|\varphi_n - A_\infty r_1^n - B_\infty r_2^n| \leqslant K_1 \sum_{k=n-1}^\infty |a_k - a|, \qquad n \in \mathbb{N}, \tag{35}$$

holds on Δ, *where the constant* K_1 *is independent of* $z \in \Delta$ *and* n (*but depends on the choice of* Δ).

(2) *If* $\sum_{n=0}^\infty |a_n - a| < \infty$ *and* $\Delta = \bar{\Delta} \subset \Delta_\alpha^o$, *then there exist two functions* $C_\infty \in C(\Delta_\alpha^o)$ *and* $D_\infty \in C(\Delta_\alpha^o)$ *such that*

$$|\varphi_n - C_\infty \hat{\varphi}_n - D_\infty \hat{\psi}_n| \leqslant K_2 \sum_{k=n-1}^\infty |a_k - a|, \qquad n \in \mathbb{N}, \tag{36}$$

holds on Δ, *where the constant* K_2 *is independent of* $z \in \Delta$ *and* n (*but depends on the choice of* Δ).[5]

[5] In fact, if the two sets Δ in (1) and (2) are the same then $K_1 = K_2$.

(3) *If* $\sum_{n=0}^{\infty} n\,|a_n - a| < \infty$, *then there exist two functions* A_∞ *and* B_∞ *satisfying* $(r_2 - r_1)\,A_\infty \in \mathrm{C}(\Delta_\alpha)$ *and* $(r_2 - r_1)\,B_\infty \in \mathrm{C}(\Delta_\alpha)$, *such that*

$$|r_2 - r_1|\,|\varphi_n - A_\infty r_1^n - B_\infty r_2^n| \leqslant K_3 \sum_{k=n-1}^{\infty} k\,|a_k - a|, \qquad n \in \mathbb{N}, \quad (37)$$

holds on Δ_α, *where the constant* K_3 *is independent of* $z \in \Delta_\alpha$ *and* n.

Proof. Fixing $\varepsilon > 0$, it is sufficient to prove (35)–(37) for $n \geqslant n_0 \stackrel{\text{def}}{=} N_1(\varepsilon)$ (cf. (24)).

Proof of (1). First note that, given a compact $\Delta \subset \Delta_\alpha^o$, there exist $d > 0$ and $v > 0$, such that $d_n \leqslant d$ and $v_{kn} \leqslant v v_k$ for $n \in \mathbb{N}$ (cf. (32), (33), and (34)). Thus we can apply Proposition 2 to (31) and use (24) to obtain that $\sup_{z \in \Delta,\, n \geqslant n_0} |\varphi_n(z)| < \infty$. Hence, $\sup_{z \in \Delta,\, n \in \mathbb{Z}^+} |\varphi_n(z)| < \infty$ as well. For $n \in \mathbb{N} \cup \{\infty\}$ define A_n and B_n by

$$\begin{aligned} A_n \stackrel{\text{def}}{=} &- \sum_{k=n_0+1}^{n-1} \frac{r_1^{-k}}{r_2 - r_1} (r_{1,k-1} + r_{2,k-1} - r_1 - r_2)\,\varphi_k \\ &+ \sum_{k=n_0}^{n-2} \frac{r_1^{-k-1}}{r_2 - r_1} (r_{1,k} r_{2,k} - r_1 r_2)\,\varphi_k + \frac{r_2 r_1^{-n_0}}{r_2 - r_1} \varphi_{n_0} - \frac{r_1^{-n_0}}{r_2 - r_1} \varphi_{n_0+1} \end{aligned}$$

and

$$\begin{aligned} B_n \stackrel{\text{def}}{=} &\sum_{k=n_0+1}^{n-1} \frac{r_2^{-k}}{r_2 - r_1} (r_{1,k-1} + r_{2,k-1} - r_1 - r_2)\,\varphi_k \\ &- \sum_{k=n_0}^{n-2} \frac{r_2^{-k-1}}{r_2 - r_1} (r_{1,k} r_{2,k} - r_1 r_2)\,\varphi_k - \frac{r_1 r_2^{-n_0}}{r_2 - r_1} \varphi_{n_0} + \frac{r_2^{-n_0}}{r_2 - r_1} \varphi_{n_0+1}. \end{aligned}$$

Then, by (24), $\lim_{n\to\infty} A_n = A_\infty$ and $\lim_{n\to\infty} B_n = B_\infty$. Since $\varphi_n = A_n r_1^n + B_n r_2^n$ (cf. (30)), (35) follows from (24) and

$$\varphi_n - A_\infty r_1^n - B_\infty r_2^n = (A_n - A_\infty)\,r_1^n + (B_n - B_\infty)\,r_2^n\ .$$

Proof of (2). The equivalence of (35) and (36) follows from the fact that both $\{r_1^n, r_2^n\}_0^\infty$ and $\{\hat{\varphi}_n, \hat{\psi}_n\}_0^\infty$ are bases for the solutions of (19).

Proof of (3). For the entire arc Δ_α there exist $d > 0$ and $v > 0$, such that $d_n \leqslant nd$ and $v_{kn} \leqslant nvv_k$ for $n \in \mathbb{N}$ (cf. (32), (33), and (34)). Rewrite (31) as

$$\left|\frac{\varphi_n(z)}{n}\right| \leqslant \frac{d_n(z)}{n} + \sum_{k=n_0}^{n-1} \frac{k}{n} v_{kn}(z) \left|\frac{\varphi_k(z)}{k}\right|, \qquad |z| = 1, \qquad n \geqslant n_0,$$

and then apply Proposition 2 to obtain $\sup_{z \in \Delta_\alpha,\, n \in \mathbb{N}} |\varphi_n(z)/n| < \infty$ which can be used to complete the proof similarly to that of part (1). ∎

Remark 6. The inequalities

$$\sup_{z \in \Delta,\, n \geqslant n_0} |\varphi_n(z)| < \infty$$

and

$$\sup_{z \in \Delta_\alpha,\, n \in \mathbb{N}} |\varphi_n(z)/n| < \infty$$

which are valid under the assumptions $\sum_{n=0}^{\infty} |a_n - a| < \infty$ and $\sum_{n=0}^{\infty} n\, |a_n - a| < \infty$, respectively, are crucial in the proof of Theorem 5. They were first proved in [12, Theorem 14, p. 414]. Asymptotics for the orthogonal polynomials in the case of asymptotically periodic coefficients, which generalize (35) and (36), were obtained in [24, Corollary 3.1, p. 347].

To determine the asymptotic behavior of the orthonormal polynomials $\{\varphi_n\}_0^\infty$ at the endpoints $z = e^{\pm i\alpha}$ of the arc, we apply Proposition 4. This approach works only for the case $r_1 = r_2$ (cf. (23)) and cannot be used to prove either (35) or (37).

THEOREM 7. *Let* $|a_n| < 1$ *for* $n \in \mathbb{N}$, $0 < |a| < 1$, $\sin(\alpha/2) \stackrel{\text{def}}{=} |a|$ *with* $\alpha \in (0, \pi)$, *and let* $\{\varphi_n\}_0^\infty$ *be a solution of* (7) (*cf.* (18)). *If* $\sum_{n=0}^{\infty} n\, |a_n - a| < \infty$, *then there exist four complex numbers* c_1, d_1, c_2, *and* d_2, *such that* $|c_1| + |d_1| > 0$, $|c_2| + |d_2| > 0$, *and*

$$\lim_{n \to \infty} \frac{\varphi_n(e^{i\alpha})\, e^{-i(\alpha/2)n}}{c_1 n + d_1} = 1 \qquad \text{and} \qquad \lim_{n \to \infty} \frac{\varphi_n(e^{-i\alpha})\, e^{i(\alpha/2)n}}{c_2 n + d_2} = 1. \tag{38}$$

Proof. Apply Proposition 4 with $f(n) \stackrel{\text{def}}{=} \varphi_n(e^{\pm i\alpha}) \exp(\mp i(\alpha/2)n)$ for $n \geqslant n_0$ (cf. (6)). ∎

Remark 8. There is a somewhat different way to prove Theorem 7. One could follow the proof in [3, Theorem 4, p. 377] after replacing $p_n(1)$ by $\varphi_n(e^{i\alpha})\, e^{-i(\alpha/2)n}$. Reference [3, Theorem 4] was generalized in [14, Theorem 3, p. 247] by considering it in the more general context of linear difference equations.

Remark 9. It is possible to rewrite (38) in the spirit of (36) (cf. [14, Theorem 4, pp. 247–248]).

Remarks 10. Analogous results can be derived from Theorems 5 and 7 by replacing $\{\varphi_n\}_0^\infty$ either by $\{\varphi_n^*\}_0^\infty$, or by $\{\psi_n\}_0^\infty$, or by $\{\psi_n^*\}_0^\infty$. For instance, (38) can be replaced by

$$\lim_{n \to \infty} \frac{\varphi_n^*(e^{i\alpha})\, e^{-i(\alpha/2)n}}{\bar{c}_1\, n + \bar{d}_1} = 1 \qquad \text{and} \qquad \lim_{n \to \infty} \frac{\varphi_n^*(e^{-i\alpha})\, e^{i(\alpha/2)n}}{\bar{c}_2\, n + \bar{d}_2} = 1,$$

and, similarly to (37), one can write

$$|r_2 - r_1|\, |\varphi_n^* - \bar{A}_\infty r_1^n - \bar{B}_\infty r_2^n| \leqslant K_4 \sum_{k=n-1}^{\infty} k\, |a_k - a|, \qquad n \in \mathbb{N}.$$

The corresponding formulas for $\{\psi_n\}_0^\infty$ and $\{\psi_n\}_0^\infty$ are almost identical to those for $\{\varphi_n\}_0^\infty$ and $\{\varphi_n^*\}_0^\infty$, respectively. More specifically, one only needs to replace a by $-a$, $\{a_n\}_0^\infty$ by $\{-a_n\}_0^\infty$, and the φ's by ψ's in the formulas involving $\{\varphi_n\}_0^\infty$ and $\{\varphi_n\}_0^\infty$. Other immediate extensions apply to the kth associated polynomials $\{\varphi_n^{(k)}\}_{n=0}^\infty$, $\{\psi_n^{(k)}\}_{n=0}^\infty$, and their *-transforms (cf. [22, Theorem 3.1, p. 176; 26, Sect. 4]).

5. MASS POINTS OF THE MEASURE

In this section we are to relax the conditions we have imposed on the reflection coefficients $\{a_n\}_0^\infty$. Although we will not be able to obtain asymptotic formulas for $\{\varphi_n\}_0^\infty$, we will still find useful information about the orthogonality measure. Such type of results come from [18, Theorem 2, p. 565]. However, the technique used here comes from the one used in [19, 20] (see, e.g., the idea of reducing the original second order difference equation to a first order one by introducing a new variable in the proof of [20, Theorem, p. 35], and formulas (7), (8), and (9) in [20, pp. 35–36]). It is possible to view the next theorem in the more general context of second order linear difference equations which provides a common platform for all of the above mentioned results (cf. [25, Chap. V]).

We remind the reader of two facts about the structure of μ and $\operatorname{supp}(\mu)$ (see (3) and (4) for the notation). Let $a \in \mathbb{C}$ with $0 < |a| < 1$. First, if $\lim_{n\to\infty} \Phi_n(\mu, 0) = a$, then $\Delta_\alpha \subseteq \operatorname{supp}(\mu)$ and $\operatorname{supp}(\mu) \setminus \Delta_\beta$ is finite for every $0 < \beta < \alpha$ (cf. [7, Theorem 1′, p. 205; 12, Theorem 3, p. 401]). Second, if $\sum_{k=1}^\infty |\Phi_k(\mu, 0) - a| < \infty$, then μ is absolutely continuous on the open circular arc Δ_α^o (cf. [12, Theorem 12, p. 410; 24, Theorem 4.1, p. 248]).

Recall that z_1 and z_2 are the zeros of (10), that is,

$$z_1 = \frac{z + 1 + \sqrt{(z - e^{i\alpha})(z - e^{-i\alpha})}}{2} \quad \text{and} \quad z_2 = \frac{z + 1 - \sqrt{(z - e^{i\alpha})(z - e^{-i\alpha})}}{2}.$$

THEOREM 11. *Let $z \in \Delta_\alpha^o$ and $0 < p < \infty$. Let $|a_n| < 1$ for $n \in \mathbb{N}$, $0 < |a| < 1$, $\sin(\alpha/2) \overset{\text{def}}{=} |a|$ with $\alpha \in (0, \pi)$, $\lim_{n\to\infty} a_n = a$, and let $\{\varphi_n\}_0^\infty$ be a solution of* (7) (*cf.* (18)). *If for some $\delta > 0$*

$$\sum_{n=0}^{\infty} \exp\left\{ \frac{-(17+\delta)\, p \sum_{k=0}^n |a_k - a|}{|z_1 - z_2|\, |a| \sqrt{1 - |a|^2}} \right\} = \infty \tag{39}$$

then

$$\sum_{n=0}^{\infty} |\varphi_n(z)|^p = \infty. \tag{40}$$

Proof. In what follows, let $\varepsilon > 0$ and pick $\Delta = \bar{\Delta} \subset \Delta_\alpha^o$ so that $z \in \Delta$. Let $n_1 \geqslant \max(N_0, N_1(\varepsilon), N_2(\varepsilon, \Delta))$ (cf. Section 3) so that $r_{1,n} \neq r_{2,n}$ holds for $n \geqslant n_1$ (cf. (27)). We decompose Eq. (18) in the following manner. Put

$$\vartheta_{1,1}^{(n)} \stackrel{\text{def}}{=} \varphi_{n+1} - r_1 \varphi_n \qquad \text{and} \qquad \vartheta_{1,2}^{(n)} \stackrel{\text{def}}{=} \varphi_{n+1} - r_2 \varphi_n, \tag{41}$$

and

$$\vartheta_{2,1}^{(n)} \stackrel{\text{def}}{=} \vartheta_{1,1}^{(n+1)} - r_2 \vartheta_{1,1}^{(n)} \qquad \text{and} \qquad \vartheta_{2,2}^{(n)} \stackrel{\text{def}}{=} \vartheta_{1,2}^{(n+1)} - r_1 \vartheta_{1,2}^{(n)}. \tag{42}$$

Then, by (41), (42), and (18),

$$\begin{aligned} \vartheta_{2,1}^{(n)} = \vartheta_{2,2}^{(n)} &= \varphi_{n+2} - (r_1 + r_2)\,\varphi_{n+1} + r_1 r_2 \varphi_n \\ &= (r_{1,n} + r_{2,n} - r_1 - r_2)\,\varphi_{n+1} - (r_{1,n} r_{2,n} - r_1 r_2)\,\varphi_n, \end{aligned}$$

so that

$$|\vartheta_{2,1}^{(n)}| + |\vartheta_{2,2}^{(n)}| \leqslant 2\,|r_{1,n} + r_{2,n} - r_1 - r_2|\,|\varphi_{n+1}| + 2\,|r_{1,n} r_{2,n} - r_1 r_2|\,|\varphi_n|. \tag{43}$$

Using (18), $|\varphi_n|$ can be estimated by

$$|\varphi_n| \leqslant \frac{|r_{1,n} + r_{2,n}|\,|\varphi_{n+1}|}{|r_{1,n} r_{2,n}|} + \frac{|\varphi_{n+2}|}{|r_{1,n} r_{2,n}|}, \qquad n \geqslant n_1. \tag{44}$$

Next, it follows from (41) that

$$\varphi_{n+1} = \frac{\vartheta_{1,2}^{(n+1)} - \vartheta_{1,1}^{(n+1)}}{r_1 - r_2} \qquad \text{and} \qquad \varphi_{n+2} = \frac{r_1 \vartheta_{1,2}^{(n+1)} - r_2 \vartheta_{1,1}^{(n+1)}}{r_1 - r_2},$$

and, hence,

$$|\varphi_{n+1}| \leqslant \frac{|\vartheta_{1,1}^{(n+1)}| + |\vartheta_{1,2}^{(n+1)}|}{|r_1 - r_2|} \quad \text{and} \quad |\varphi_{n+2}| \leqslant \frac{|\vartheta_{1,1}^{(n+1)}| + |\vartheta_{1,2}^{(n+1)}|}{|r_1 - r_2|}, \quad n \geqslant n_1. \tag{45}$$

Combining (43), (44), and (45), we obtain

$$|\vartheta_{2,1}^{(n)}|+|\vartheta_{2,2}^{(n)}|\leqslant\frac{2}{|r_1-r_2|}\left(|\vartheta_{1,1}^{(n+1)}|+|\vartheta_{1,2}^{(n+1)}|\right)\times\left\{|r_{1,n}+r_{2,n}-r_1-r_2|+|r_{1,n}r_{2,n}-r_1r_2|\frac{|r_{1,n}+r_{2,n}|}{|r_{1,n}r_{2,n}|}+\frac{|r_{1,n}r_{2,n}-r_1r_2|}{|r_{1,n}r_{2,n}|}\right\} \tag{46}$$

for $n\geqslant n_1$. Thus, using (24) with the previously fixed $\varepsilon>0$, we can choose $n_2\geqslant n_1$ so that

$$|\vartheta_{2,1}^{(n)}|+|\vartheta_{2,2}^{(n)}|\leqslant\frac{2}{|r_1-r_2|}\left(|\vartheta_{1,1}^{(n+1)}|+|\vartheta_{1,2}^{(n+1)}|\right)e_n,\qquad n\geqslant n_2,$$

where

$$e_n\stackrel{\text{def}}{=}(E_1+3E_3+5\varepsilon)\,|a_{n+1}-a|+(E_2+3E_3+5\varepsilon)\,|a_{n+2}-a|. \tag{47}$$

From (42) it follows that

$$|\vartheta_{2,1}^{(n)}|\geqslant|\vartheta_{1,1}^{(n)}|-|\vartheta_{1,1}^{(n+1)}|\qquad\text{and}\qquad|\vartheta_{2,2}^{(n)}|\geqslant|\vartheta_{1,2}^{(n)}|-|\vartheta_{1,2}^{(n+1)}|.$$

Thus, by (46),

$$\begin{aligned}|\vartheta_{1,1}^{(n)}|+|\vartheta_{1,2}^{(n)}|&\leqslant\left(|\vartheta_{1,1}^{(n+1)}|+|\vartheta_{1,2}^{(n+1)}|\right)\left\{1+\frac{2e_n}{|r_1-r_2|}\right\}\\&\leqslant\left(|\vartheta_{1,1}^{(n+1)}|+|\vartheta_{1,2}^{(n+1)}|\right)\exp\left\{\frac{2e_n}{|r_1-r_2|}\right\},\qquad n\geqslant n_2.\end{aligned} \tag{48}$$

Iterating (48), we obtain

$$|\vartheta_{1,1}^{(n+1)}|+|\vartheta_{1,2}^{(n+1)}|\geqslant\left(|\vartheta_{1,1}^{(n_2)}|+|\vartheta_{1,2}^{(n_2)}|\right)\exp\left\{\frac{-2\sum_{k=n_2}^{n}e_k}{|r_1-r_2|}\right\},\qquad n\geqslant n_2. \tag{49}$$

Here $|\vartheta_{1,1}^{(n_2)}|+|\vartheta_{1,2}^{(n_2)}|>0$ since otherwise, from (41), $\varphi_{n_2}=\varphi_{n_2+1}=0$, and then (8) implies that $a_{n_2+1}=0$ as opposed to the choice of n_2.[6]

By (41),

$$|\vartheta_{1,1}^{(n+1)}|\leqslant|\varphi_{n+1}|+|\varphi_{n+2}|\qquad\text{and}\qquad|\vartheta_{1,2}^{(n+1)}|\leqslant|\varphi_{n+1}|+|\varphi_{n+2}|.$$

[6] In fact, there is no need to use (8). Since all the zeros of all φ_n's are in the open unit disk (cf. [27, Theorem 11.4.1, p. 292]), it follows from (41) directly that $|\vartheta_{1,1}^{(n)}|+|\vartheta_{1,2}^{(n)}|>0$ for $n\in\mathbb{N}$.

Thus, by (49),

$$|\varphi_{n+1}|+|\varphi_{n+2}|\geqslant\frac{|\vartheta_{1,1}^{(n_2)}|+|\vartheta_{1,2}^{(n_2)}|}{2}\exp\left\{\frac{-2\sum_{k=n_2}^{n}e_k}{|r_1-r_2|}\right\}.$$

Given $p>0$, let $c_p\stackrel{\text{def}}{=}\max(1,2^{p-1})$. Then

$$\begin{aligned}c_p(|\varphi_{n+1}|^p+|\varphi_{n+2}|^p)&\geqslant\left(\frac{|\vartheta_{1,1}^{(n_2)}|+|\vartheta_{1,2}^{(n_2)}|}{2}\right)^p\exp\left\{\frac{-2p\sum_{k=0}^{n}e_k}{|r_1-r_2|}\right\}\\&\geqslant\left(\frac{|\vartheta_{1,1}^{(n_2)}|+|\vartheta_{1,2}^{(n_2)}|}{2}\right)^p\exp\left\{\frac{-2p\sqrt{1-|a|^2}\,E\sum_{k=1}^{n+2}|a_k-a|}{|z_1-z_2|}\right\},\\&\qquad n\geqslant n_2,\end{aligned}$$

where $E=E(a,\varepsilon)\stackrel{\text{def}}{=}E_1+E_2+6E_3+10\varepsilon$ (cf. (47)). Now the theorem follows from

$$\sum_{n=n_2+1}^{\infty}|\varphi_n|^p\geqslant\frac{(|\vartheta_{1,1}^{(n_2)}|+|\vartheta_{1,2}^{(n_2)}|)^p}{2^{p+1}c_p}\sum_{n=n_2+1}^{\infty}\exp\left\{\frac{-2p\sqrt{1-|a|^2}\,E\sum_{k=1}^{n+1}|a_k-a|}{|z_1-z_2|}\right\},$$

where, by (26), the constant E is given by

$$E\stackrel{\text{def}}{=}\frac{2}{|a|\,(1-|a|^2)}(3+|a|^2+\sqrt{1-|a|^2})+10\varepsilon,$$

and, since $x+\sqrt{1-x}\leqslant 5/4$ for $x\in[0,1]$,

$$2\sqrt{1-|a|^2}\,E\leqslant\frac{17+\delta}{|a|\sqrt{1-|a|^2}},$$

where $\delta=20\sqrt{1-|a|^2}\,\varepsilon$. ∎

COROLLARY 12. *If the conditions of Theorem* 11 *hold with* $p=2$ *in* (39), *then the orthogonality measure* μ *corresponding to* $\{\varphi_n\}_0^\infty$ *has no mass point at that particular point* $z\in\Delta_\alpha^o$.

Proof. By (40), $\sum_{n=0}^{\infty} |\varphi_n(\mu, z)|^2 = \infty$. Hence the corollary follows from the well known formula

$$\sum_{n=0}^{\infty} |\varphi_n(\mu, z)|^2 = \frac{1}{\mu(\{\vartheta\})}, \qquad z = e^{i\vartheta}, \tag{50}$$

(cf. [16, formula (7) on p. 453 and its proof on pp. 444–445]). ∎

COROLLARY 13. *Let $|a_n| < 1$ for $n \in \mathbb{N}$, $0 < |a| < 1$, $\sin(\alpha/2) \stackrel{\text{def}}{=} |a|$ with $\alpha \in (0, \pi)$, $\lim_{n\to\infty} a_n = a$, and let $\{\varphi_n\}_0^\infty$ be a solution of* (7) (*cf.* (18)). *If for every $\tau \in \mathbb{R}$*

$$\sum_{n=0}^{\infty} \exp\left\{\tau \sum_{k=0}^{n} |a_k - a|\right\} = \infty, \tag{51}$$

then for every $z \in \Delta_\alpha^o$ and $p > 0$ we have $\{\varphi_n(z)\}_{n=0}^\infty \notin \ell_p$. In particular, the corresponding orthogonality measure μ has no mass points in Δ_α^o.

Remark 14. If either $|a_n - a| = o(1/n)$ or $\sum_{n=0}^\infty |a_n - a| < \infty$, then (51) holds.

Remark 15. One can eliminate the use of z_1 and z_2 from (39) in the following way. Let $z = e^{i\vartheta}$ with $\alpha < \vartheta \leqslant \pi$. Then we have

$$\begin{aligned} |z_1 - z_2| &= \left|\sqrt{(z - e^{i\alpha})(z - e^{-i\alpha})}\right| = 2\sqrt{\sin\frac{\vartheta - \alpha}{2}\sin\frac{\vartheta + \alpha}{2}} \\ &\geqslant \frac{2}{\sqrt{\pi}}\sqrt{(1 - |a|^2)^{1/2}\min(1, 2|a|)} \times |\vartheta - \alpha|^{1/2}, \end{aligned}$$

where we used $|\sin((\vartheta - \alpha)/2)| \geqslant |\vartheta - \alpha|/\pi$ and $|\sin((\vartheta + \alpha)/2)| \geqslant \min(\sin\alpha, \sin\frac{\pi+\alpha}{2})$ for $\alpha < \vartheta \leqslant \pi$. Another possible inequality is given by

$$|z_1 - z_2| = 2\sqrt{\sin\frac{\vartheta - \alpha}{2}\sin\frac{\vartheta + \alpha}{2}} > 2\left|\sin\frac{\vartheta - \alpha}{2}\right| \geqslant \frac{2}{\pi}|\vartheta - \alpha|,$$

where we used $|\sin((\vartheta + \alpha)/2)| > |\sin((\vartheta - \alpha)/2)|$ for $\alpha < \vartheta \leqslant \pi$.

EXAMPLE 16. One cannot replace the condition $|a_n - a| = o(1/n)$ by $|a_n - a| = O(1/n)$ in Corollary 13, since there are measures μ and corresponding orthogonal polynomials $\{\varphi_n\}_0^\infty$ with reflection coefficients $\{a_n\}_0^\infty$, such that $\lim_{n\to\infty} n\,|a_n - a| > 0$, and μ has a mass point in Δ_α^o. Indeed, let $a \stackrel{\text{def}}{=} 1 + i/2$, and consider the Geronimus polynomials $\{\hat{\varphi}_n\}_0^\infty$ along with their measure of orthogonality $\hat{\mu}_a$ (cf. Section 2). We construct a new measure by adding a mass point at a fixed $z_0 \in \Delta_\alpha^o$, and then renormalizing the resulting measure. More specifically, let $\mu = \mu_{a, z_0} \stackrel{\text{def}}{=} (\hat{\mu}_a + \delta_{z_0})/2$. Denote by $\{\varphi_n\}_0^\infty$ and $\{\Phi_n\}_0^\infty$ the sequences of orthonormal and monic orthogonal polynomials with respect to μ, respectively. The relation between Φ_n and Φ_n is given by

$$\Phi_n(z)=\hat{\Phi}_n(z)-\frac{\hat{\Phi}_n(z_0)\,\hat{K}_{n-1}(z,z_0)}{1+\hat{K}_{n-1}(z_0,z_0)},$$

where

$$\hat{K}_m(z,u)\stackrel{\text{def}}{=}\sum_{j=0}^{m}\hat{\varphi}_j(z)\,\overline{\hat{\varphi}_j(u)}, \tag{52}$$

(cf. [2, p. 525; 11, formula (2.8), p. 36]). Putting $z=0$ and taking into account $\hat{K}_{n-1}(0,z_0)=\hat{\kappa}_{n-1}\,\overline{\hat{\varphi}^*_{n-1}(z_0)}$ (cf. [8, Chap. 1, formula (1.9)]), we obtain

$$a_n=a-\frac{\hat{\kappa}_{n-1}}{\hat{\kappa}_n}\,\frac{\hat{\varphi}_n(z_0)\,\overline{\hat{\varphi}^*_{n-1}(z_0)}}{1+\hat{K}_{n-1}(z_0,z_0)},$$

so that

$$|a_n-a|=(1-|a|^2)^{1/2}\,\frac{|\hat{\varphi}_n(z_0)\,\hat{\varphi}_{n-1}(z_0)|}{1+\hat{K}_{n-1}(z_0,z_0)}. \tag{53}$$

Let $z_0\stackrel{\text{def}}{=}-1$. It follows from (16) that

$$\hat{\varphi}_n(-1)\,\overline{\hat{\varphi}_{n-1}(-1)}=(-1)^n\exp\left\{(-1)^{n+1}\frac{i\alpha}{2}\right\}.$$

Hence, by (53) and (17),

$$\begin{aligned}\lim_{n\to\infty} n\,|a_n-a|&=\lim_{n\to\infty}\frac{n(1-|a|^2)^{1/2}}{1+\hat{K}_{n-1}(-1,-1)}\\ &=\cos(\alpha/2)\,\frac{(1-|a|^2)^{1/2}}{|1-a|}=\frac{1}{\sqrt{2}}>0. \qquad \blacksquare\end{aligned}$$

The situation concerning mass points at the endpoints of the arc is more delicate. The following statement is a direct consequence of Theorem 7 and (50).

THEOREM 17. *Let* $|a_n|<1$ *for* $n\in\mathbb{N}$, $0<|a|<1$, $\sin(\alpha/2)\stackrel{\text{def}}{=}|a|$ *with* $\alpha\in(0,\pi)$, *and let* $\{\varphi_n\}_0^\infty$ *be a solution of* (7) (*cf.* (18)). *If* $\sum_{n=0}^\infty n\,|a_n-a|<\infty$, *then orthogonality measure* μ *has no mass points at the endpoints of* Δ_α (*cf.* (4)).

6. SEQUENCES OF BOUNDED VARIATION

Another Solution Formula for the Perturbed Equation. A possible relaxation of the condition $\sum_{n=0}^{\infty} |a_n - a| < \infty$ is to assume that $\{a_n\}_0^\infty$ is of bounded variation, that is,

$$\sum_{n=0}^{\infty} |a_{n+1} - a_n| < \infty. \tag{54}$$

Using (28), the latter property can also be described in terms of the roots $r_{j,n}$ of the characteristic equation (20). To be able to use (54), we rewrite (19) as

$$\hat{\varphi}_{n+2} - r_1 \hat{\varphi}_{n+1} = r_2(\hat{\varphi}_{n+1} - r_1 \hat{\varphi}_n). \tag{55}$$

In what follows, let $\varepsilon > 0$ and $\Delta = \bar{\Delta} \subset \Delta_\alpha^o$. Let $n_1 \geqslant \max(N_0, N_1(\varepsilon), N_2(\varepsilon, \Delta))$ (cf. Section 3) and $z \in \Delta$ so that $r_{1,n} \neq r_{2,n}$ holds for $n \geqslant n_1$ (cf. (27)). For the perturbed equation (18) we follow the technique introduced in [13, p. 614]. We set

$$g_n \stackrel{\text{def}}{=} \varphi_{n+1} - r_{1,n} \varphi_n$$

so that (18) becomes

$$g_{n+1} - r_{2,n} g_n = (r_{1,n} - r_{1,n+1}) \varphi_{n+1}. \tag{56}$$

This may be solved as follows (cf. [13; 15; 29; 28, p. 450] for a similar analysis of orthogonal polynomials on the real line). Let

$$G_{n_1} \stackrel{\text{def}}{=} g_{n_1} \qquad \text{and} \qquad G_n \stackrel{\text{def}}{=} g_n \Big/ \prod_{k=n_1}^{n-1} r_{2,k}, \qquad n \geqslant n_1 + 1.$$

Then

$$G_{n+1} - G_n = (r_{1,n} - r_{1,n+1}) \varphi_{n+1} \Big/ \prod_{k=n_1}^{n} r_{2,k}$$

so that

$$G_n = G_{n_1} + \sum_{k=n_1}^{n-1} \frac{(r_{1,k} - r_{1,k+1}) \varphi_{k+1}}{\prod_{j=n_1}^{k} r_{2,j}}, \qquad n \geqslant n_1.$$

The symmetric role of r_1 and r_2 in (55) suggests introducing

$$h_n \stackrel{\text{def}}{=} \varphi_{n+1} - r_{2,n} \varphi_n$$

and

$$H_{n_1} \stackrel{\text{def}}{=} h_{n_1} \qquad \text{and} \qquad H_n \stackrel{\text{def}}{=} h_n \Big/ \prod_{k=n_1}^{n-1} r_{1,k}, \qquad n \geqslant n_1 + 1.$$

Then

$$H_n = H_{n_1} + \sum_{k=n_1}^{n-1} \frac{(r_{2,k} - r_{2,k+1})\,\varphi_{k+1}}{\prod_{j=n_1}^{k} r_{1,j}}, \qquad n \geqslant n_1.$$

Now we can derive an implicit solution formula for φ_n using

$$\varphi_n = \frac{h_n - g_n}{r_{1,n} - r_{2,n}} \qquad \text{and} \qquad \varphi_{n+1} = \frac{r_{1,n} h_n - r_{2,n} g_n}{r_{1,n} - r_{2,n}}$$

which we write as

$$\begin{aligned} \varphi_n = {} & \frac{1}{r_{1,n} - r_{2,n}} \left(H_{n_1} + \sum_{k=n_1}^{n-1} \frac{(r_{2,k} - r_{2,k+1})\,\varphi_{k+1}}{\prod_{j=n_1}^{k} r_{1,j}} \right) \prod_{k=n_1}^{n-1} r_{1,k} \\ & - \frac{1}{r_{1,n} - r_{2,n}} \left(G_{n_1} + \sum_{k=n_1}^{n-1} \frac{(r_{1,k} - r_{1,k+1})\,\varphi_{k+1}}{\prod_{j=n_1}^{k} r_{2,j}} \right) \prod_{k=n_1}^{n-1} r_{2,k}, \qquad n \geqslant n_1. \end{aligned} \tag{57}$$

In this section, our first result is about upper bounds and asymptotics for the orthonormal polynomials $\{\varphi_n\}_0^\infty$ whose reflection coefficients satisfy (54).

THEOREM 18. *Let* $|a_n| < 1$ *for* $n \in \mathbb{N}$, $0 < |a| < 1$, $\sin(\alpha/2) \stackrel{\text{def}}{=} |a|$ *with* $\alpha \in (0, \pi)$, *let* $\{\varphi_n\}_0^\infty$ *be a solution of* (7) (*cf.* (18)), *and let* $\Delta = \bar{\Delta} \subset \Delta_\alpha^o$. *Assume that* $\lim_{n\to\infty} a_n = a$, $\sum_{n=0}^\infty |a_{n+1} - a_n| < \infty$, *and* $\Omega \stackrel{\text{def}}{=} \max\{\Omega_1, \Omega_2\} < \infty$ *with*

$$\Omega_1 \stackrel{\text{def}}{=} \sup_{z\in\Delta} \sup_{\sigma > N_0} \prod_{j=N_0+1}^{\sigma} |r_{1,j}| < \infty$$

and

$$\Omega_2 \stackrel{\text{def}}{=} \sup_{z\in\Delta} \sup_{\sigma > N_0} \prod_{j=N_0+1}^{\sigma} |r_{2,j}| < \infty, \tag{58}$$

where N_0 *is defined by* (6). *Then*

$$\sup_{z \in \Delta} \sup_{n \in \mathbb{Z}^+} |\varphi_n(z)| < \infty \tag{59}$$

and there exist a constant K_5 *independent of* $z \in \Delta$ *and* n (*but it depends on the choice of* Δ) *and two functions* $H_\infty \in \mathrm{C}(\Delta_\alpha^o)$ *and* $G_\infty \in \mathrm{C}(\Delta_\alpha^o)$, *such that*

$$\left|(r_{1,n} - r_{2,n})\, \varphi_n - H_\infty \prod_{k=N_0}^{n-1} r_{1,k} + G_\infty \prod_{k=N_0}^{n-1} r_{2,k}\right|$$

$$\leqslant K_5 \sum_{k=n+1}^{\infty} |a_{k+1} - a_k|, \qquad n > N_0, \tag{60}$$

in Δ.

Proof. Let $z \in \Delta$, $\varepsilon > 0$, and let $0 < \gamma < 1$ and $n_1 \geqslant \max(N_0, N_1(\varepsilon), N_2(\varepsilon, \Delta))$ (cf. Section 3) be such that

$$|r_{1,n} - r_{2,n}| \geqslant \gamma,$$

$$\Omega(|r_{1,n} - r_{1,n+1}| + |r_{2,n} - r_{2,n+1}|) < 1 - \gamma, \qquad z \in \Delta, \qquad n > n_1. \tag{61}$$

Choosing γ and n_1 in (61) is possible because we have

$$\lim_{n \to \infty} |r_{1,n} - r_{2,n}| = |r_1 - r_2| = \left|\sqrt{\frac{(z-1)^2 + 4z\,|a|^2}{1 - |a|^2}}\right|,$$

$\lim_{n \to \infty} |r_{1,n} - r_{1,n+1}| = 0$, and $\lim_{n \to \infty} |r_{2,n} - r_{2,n+1}| = 0$ uniformly in Δ (cf. (20)). Clearly, instead of (59), it is sufficient to prove

$$\sup_{z \in \Delta} \sup_{n \geqslant n_1} |\varphi_n(z)| < \infty. \tag{62}$$

It follows from (57) and the left-hand side of (61) that

$$|\varphi_n| \leqslant \Omega \gamma^{-1}(|H_{n_1}| + |G_{n_1}|)$$

$$+ \sum_{k=n_1}^{n-1} [\Omega(|r_{2,k} - r_{2,k+1}| + |r_{1,k} - r_{1,k+1}|)\, |\varphi_{k+1}|] \tag{63}$$

for $n \geqslant n_1$. Now use Corollary 3 applied to (63) (cf. right-hand side of (61)) and (28) to obtain (62).

Having proved (59), now we consider (60). Let again $z \in \varDelta$, $\varepsilon > 0$, and let $n_1 \geqslant \max(N_0, N_1(\varepsilon), N_2(\varepsilon, \varDelta))$ (cf. Section 3). By (57) we have

$$(r_{1,n} - r_{2,n})\,\varphi_n = H_n \prod_{k=n_1}^{n-1} r_{1,k} - G_n \prod_{k=n_1}^{n-1} r_{2,k}, \qquad n \geqslant n_1,$$

where

$$G_n \overset{\text{def}}{=} G_{n_1} + \sum_{k=n_1}^{n-1} \frac{(r_{1,k} - r_{1,k+1})\,\varphi_{k+1}}{\prod_{j=n_1}^{k} r_{2,j}}$$

and

$$H_n \overset{\text{def}}{=} H_{n_1} + \sum_{k=n_1}^{n-1} \frac{(r_{2,k} - r_{2,k+1})\,\varphi_{k+1}}{\prod_{j=n_1}^{k} r_{1,j}}.$$

Using (22), (28), (58), and (59), define G_∞ and H_∞ by

$$G_\infty \overset{\text{def}}{=} \frac{\lim_{n\to\infty} G_n}{\prod_{k=N_0}^{n_1-1} r_{2,k}} \qquad \text{and} \qquad H_\infty \overset{\text{def}}{=} \frac{\lim_{n\to\infty} H_n}{\prod_{k=N_0}^{n_1-1} r_{1,k}},$$

respectively. Then

$$\begin{aligned}
(r_{1,n} - r_{2,n})\,\varphi_n - H_\infty \prod_{k=N_0}^{n-1} r_{1,k} + G_\infty \prod_{k=N_0}^{n-1} r_{2,k} \\
= \left(H_n - H_\infty \prod_{k=N_0}^{n_1-1} r_{1,k}\right) \prod_{k=n_1}^{n-1} r_{1,k} \\
- \left(G_n - G_\infty \prod_{k=N_0}^{n_1-1} r_{2,k}\right) \prod_{k=n_1}^{n-1} r_{2,k}, \qquad n \geqslant n_1,
\end{aligned}$$

so that (60) follows from (28) when $n \geqslant n_1$. When $N_0 < n < n_1$, (60) clearly holds with an appropriate choice of K_5. ∎

Remark 19. By (22), the products in (58) also satisfy

$$0 < \inf_{z\in\varDelta}\ \inf_{\sigma > N_0} \prod_{j=N_0+1}^{\sigma} |r_{1,j}| \qquad \text{and} \qquad 0 < \inf_{z\in\varDelta}\ \inf_{\sigma > N_0} \prod_{j=N_0+1}^{\sigma} |r_{2,j}|.$$

Remark 20. If $|a_n| < 1$ for $n \in \mathbb{Z}^+$ and $0 < |a| < 1$, then (58) holds whenever $\sum_{n=0}^{\infty} |a_n - a| < \infty$.

Recall that z_1 and z_2 are the zeros of the polynomial (10), that is,

$$z_1 = \frac{z + 1 + \sqrt{(z - e^{i\alpha})(z - e^{-i\alpha})}}{2} \quad \text{and} \quad z_2 = \frac{z + 1 - \sqrt{(z - e^{i\alpha})(z - e^{-i\alpha})}}{2}.$$

The next theorem is related to Theorem 11.

THEOREM 21. *Let* $z \in \Delta_\alpha^o$ *and* $0 < p < \infty$. *Let* $|a_n| < 1$ *for* $n \in \mathbb{N}$, $0 < |a| < 1$, $\sin(\alpha/2) \stackrel{\text{def}}{=} |a|$ *with* $\alpha \in (0, \pi)$, $\lim_{n \to \infty} a_n = a$, *and let* $\{\varphi_n\}_0^\infty$ *be a solution of* (7) (*cf.* (18)). *Let*

$$\omega \stackrel{\text{def}}{=} \tfrac{1}{2} \limsup_{\ell \to \infty} \inf_{n \geqslant \ell} \prod_{k=\ell}^{n} \min(|r_{1,k}|, |r_{2,k}|) > 0, \tag{64}$$

where $r_{1,n}$ *and* $r_{2,n}$ *are the roots of* (20). *If, for some* $\delta > 0$,

$$\sum_{n=0}^{\infty} \exp\left\{ \frac{-(14 + \delta)\, p \sum_{k=0}^{n} |a_{k+1} - a_k|}{|z_1 - z_2|^2\, |a|} \right\} = \infty \tag{65}$$

then

$$\sum_{n=0}^{\infty} |\varphi_n(z)|^p = \infty.$$

Proof. The proof is a modification of that of Theorem 11. In what follows, let $\varepsilon > 0$, and let $\ell_0 \geqslant \max(N_0, N_1(\varepsilon), N_2(\varepsilon, \Delta))$ (cf. Section 3) so that $r_{1,n} \neq r_{2,n}$ holds for $n \geqslant l_0$ (cf. (27)). We start with the decomposition of (18) (cf. (56)) by introducing

$$\eta_{1,1}^{(n)} \stackrel{\text{def}}{=} \varphi_{n+1} - r_{1,n}\, \varphi_n \qquad \text{and} \qquad \eta_{1,2}^{(n)} \stackrel{\text{def}}{=} \varphi_{n+1} - r_{2,n}\, \varphi_n, \tag{66}$$

and

$$\eta_{2,1}^{(n)} \stackrel{\text{def}}{=} \eta_{1,1}^{(n+1)} - r_{2,n}\, \eta_{1,1}^{(n)} \qquad \text{and} \qquad \eta_{2,2}^{(n)} \stackrel{\text{def}}{=} \eta_{1,2}^{(n+1)} - r_{1,n}\, \eta_{1,2}^{(n)}. \tag{67}$$

Then, by (66), (67), and (18),

$$\eta_{2,1}^{(n)} = (r_{1,n} - r_{1,n+1})\, \varphi_{n+1} \tag{68}$$

and

$$\eta_{2,2}^{(n)} = (r_{2,n} - r_{2,n+1})\,\varphi_{n+1}. \tag{69}$$

It follows from (66) that

$$\varphi_n = \frac{\eta_{1,2}^{(n)} - \eta_{1,1}^{(n)}}{r_{1,n} - r_{2,n}} \qquad \text{and} \qquad \varphi_{n+1} = \frac{r_{1,n}\,\eta_{1,2}^{(n)} - r_{2,n}\,\eta_{1,1}^{(n)}}{r_{1,n} - r_{2,n}}, \qquad n \geqslant \ell_0. \tag{70}$$

Combining (68) and the left-hand side of (70) applied with $n+1$ instead of n, we obtain

$$|\eta_{2,1}^{(n)}| \leqslant (|\eta_{1,1}^{(n+1)}|)\,\frac{|r_{1,n} - r_{1,n+1}|}{|r_{1,n+1} - r_{2,n+1}|}, \qquad n \geqslant \ell_0.$$

Similarly, combining (69) and the left-hand side of (70) applied with $n+1$ instead of n, we get

$$|\eta_{2,1}^{(n)}| \leqslant (|\eta_{1,1}^{(n+1)}|)\,\frac{|r_{1,n} - r_{1,n+1}|}{|r_{1,n+1} - r_{2,n+1}|}, \qquad n \geqslant \ell_0.$$

Hence,

$$|\eta_{2,1}^{(n)}| + |\eta_{2,2}^{(n)}| \leqslant (|\eta_{1,1}^{(n+1)}| + |\eta_{1,2}^{(n+1)}|)\,\frac{|r_{1,n} - r_{1,n+1}| + |r_{2,n} - r_{2,n+1}|}{|r_{1,n+1} - r_{2,n+1}|},$$

$$n \geqslant \ell_0.$$

Pick $\ell_1 \geqslant \ell_0$ in such a way that $|r_{1,n+1} - r_{2,n+1}| > |r_1 - r_2|/(1+\varepsilon)$ for $n \geqslant \ell_1$. Then we get

$$|\eta_{2,1}^{(n)}| + |\eta_{2,2}^{(n)}| \leqslant \frac{1}{|r_1 - r_2|^2}\,(|\eta_{1,1}^{(n+1)}| + |\eta_{1,2}^{(n+1)}|)\,f_n, \qquad n \geqslant \ell_1, \tag{71}$$

where, using (28) with the previously fixed $\varepsilon > 0$, we can choose

$$f_n \stackrel{\text{def}}{=} \{2(E_4 + \varepsilon)\,|a_{n+2} - a_{n+1}| + 2(E_5 + \varepsilon)\,|a_{n+3} - a_{n+2}|\}(1+\varepsilon).$$

By (67),

$$|\eta_{2,1}^{(n)}| \geqslant |r_{2,n}|\;|\eta_{1,1}^{(n)}| - |\eta_{1,1}^{(n+1)}| \geqslant \min(|r_{1,n}|, |r_{2,n}|)\;|\eta_{1,1}^{(n)}| - |\eta_{1,1}^{(n+1)}|$$

and

$$|\eta_{2,2}^{(n)}| \geqslant |r_{1,n}|\ |\eta_{1,2}^{(n)}| - |\eta_{1,2}^{(n+1)}| \geqslant \min(|r_{1,n}|, |r_{2,n}|)\ |\eta_{1,2}^{(n)}| - |\eta_{1,2}^{(n+1)}|,$$

so that by (71),

$$\begin{aligned} &\min(|r_{1,n}|, |r_{2,n}|)(|\eta_{1,1}^{(n)}| + |\eta_{1,2}^{(n)}|) \\ &\quad \leqslant (|\eta_{1,1}^{(n+1)}| + |\eta_{1,2}^{(n+1)}|) \left\{1 + \frac{f_n}{|r_1 - r_2|^2}\right\} \\ &\quad \leqslant (|\eta_{1,1}^{(n+1)}| + |\eta_{1,2}^{(n+1)}|) \exp\left\{\frac{f_n}{|r_1 - r_2|^2}\right\}, \qquad n \geqslant \ell_1. \end{aligned} \tag{72}$$

Let $\ell_2(\geqslant \ell_1) \in \mathbb{N}$ be such that $\prod_{k=\ell_2}^{n} \min(|r_{1,k}|, |r_{2,k}|) \geqslant \omega$ for $n \geqslant \ell_2$ (cf. (64)). Iterating (72) yields

$$\begin{aligned} |\eta_{1,1}^{(n+1)}| + |\eta_{1,2}^{(n+1)}| &\geqslant \prod_{k=\ell_2}^{n} \min(|r_{1,k}|, |r_{2,k}|)(|\eta_{1,1}^{(\ell_2)}| + |\eta_{1,2}^{(\ell_2)}|) \\ &\qquad \times \exp\left\{\frac{-\sum_{k=\ell_2}^{n} f_k}{|r_1 - r_2|^2}\right\} \\ &\geqslant \omega(|\eta_{1,1}^{(\ell_2)}| + |\eta_{1,2}^{(\ell_2)}|) \exp\left\{\frac{-\sum_{k=0}^{n} f_k}{|r_1 - r_2|^2}\right\}, \qquad n \geqslant \ell_2. \end{aligned} \tag{73}$$

Here $|\eta_{1,1}^{(\ell_2)}| + |\eta_{1,2}^{(\ell_2)}| > 0$ since, otherwise, by (66) (cf. (70)), $\varphi_{\ell_2} = \varphi_{\ell_2+1} = 0$, and then (8) would imply $a_{\ell_2+1} = 0$ as opposed to the choice of ℓ_2.[7]

By (66) applied with $n+1$ instead of n, $|\eta_{1,1}^{(n+1)}| \leqslant |\varphi_{n+2}| + |r_{1,n+1}|\ |\varphi_{n+1}|$ and $|\eta_{1,2}^{(n+1)}| \leqslant |\varphi_{n+2}| + |r_{2,n+1}|\ |\varphi_{n+1}|$. Note that $\lim_{n\to\infty} |r_{1,n}| = 1$ and $\lim_{n\to\infty} |r_{2,n}| = 1$. Thus, by (73), there is $\ell_3(\geqslant \ell_2) \in \mathbb{N}$ such that

$$|\varphi_{n+1}| + |\varphi_{n+2}| \geqslant \frac{|\eta_{1,1}^{(\ell_2)}| + |\eta_{1,2}^{(\ell_2)}|}{3}\, \omega \exp\left\{\frac{-\sum_{k=0}^{n} f_k}{|r_1 - r_2|^2}\right\}, \qquad n \geqslant \ell_3.$$

[7] As mentioned before, there is no need to use (8). Since all the zeros of all φ_n's are in the open unit disk (cf. [27, Theorem 11.4.1, p. 292]), it follows from (66) (cf. (70)) directly that $|\eta_{1,1}^{(n)}| + |\eta_{1,2}^{(n)}| > 0$ as long as $r_{1,n} \neq r_{2,n}$.

Given $p>0$ put $c_p \stackrel{\text{def}}{=} \max(1, 2^{p-1})$. Then

$$\begin{aligned} c_p(|\varphi_{n+1}|^p + |\varphi_{n+2}|^p) &\geqslant \left(\frac{|\eta_{1,1}^{(\ell_2)}| + |\eta_{1,2}^{(\ell_2)}|}{3}\right)^p \omega^p \exp\left\{\frac{-p\sum_{k=0}^n f_k}{|r_1-r_2|^2}\right\} \\ &\geqslant \left(\frac{|\eta_{1,1}^{(\ell_2)}| + |\eta_{1,2}^{(\ell_2)}|}{3}\right)^p \omega^p \\ &\quad \times \exp\left\{\frac{-p(1-|a|^2)\, F\sum_{k=1}^{n+2} |a_{k+1}-a_k|}{|z_1-z_2|^2}\right\}, \qquad n \geqslant \ell_3, \end{aligned}$$

where $F = F(a, \varepsilon) =^{\text{def}} (1+\varepsilon)(2E_4 + 2E_5 + 4\varepsilon)$ and we used (12) to replace $|r_1 - r_2|^{-2}$ by $(1-|a|^2)\, |z_1 - z_2|^{-2}$. Now the theorem follows from

$$\begin{aligned} \sum_{n=\ell_3+1}^{\infty} |\varphi_n|^p &\geqslant \frac{(|\eta_{1,1}^{(\ell_2)}| + |\eta_{1,2}^{(\ell_2)}|)^p}{2\, c_p 3^p}\, \omega^p \\ &\quad \times \sum_{n=\ell_3+1}^{\infty} \exp\left\{\frac{-p(1-|a|^2)\, F\sum_{k=1}^{n+1} |a_{k+1}-a_k|}{|z_1-z_2|^2}\right\}. \end{aligned}$$

For the constant F we have

$$F = \frac{8}{|a|\,(1-|a|^2)}\left(\sqrt{1-|a|^2} + |a|^2 + \frac{1}{2}\right) + O(\varepsilon),$$

and, since $x + \sqrt{1-x} \leqslant 5/4$ for $x \in [0, 1]$,

$$(1-|a|^2)\, F \leqslant \frac{14}{|a|} + O(\varepsilon),$$

giving the desired result. ∎

Remark 22. Just as in the case of Theorem 11, we have a number of corollaries. If the conditions of Theorem 21 hold with $p=2$ in (65), then the measure of orthogonality μ corresponding to $\{\varphi_n\}_0^\infty$ has no mass point at that particular point $z \in \Delta_\alpha^o$. If $\lim_{n\to\infty} a_n = a$ with $0 < |a| < 1$ and for every $\tau \in \mathbb{R}$

$$\sum_{n=0}^{\infty} \exp\left\{\tau \sum_{k=0}^{n} |a_{k+1} - a_k|\right\} = \infty, \tag{74}$$

then for every $z \in \Delta_\alpha^o$ and $p > 0$ we have $\{\varphi_n(z)\}_{n=0}^\infty \notin \ell_p$. In particular, the corresponding orthogonality measure μ has no mass points in Δ_α^o. If either $|a_{n+1} - a_n| = o(1/n)$ or $\sum_{n=0}^\infty |a_{n+1} - a_n| < \infty$, then (74) holds. The condition $|a_{n+1} - a_n| = o(1/n)$ cannot be replaced by $|a_{n+1} - a_n| = O(1/n)$ since the polynomials given in Example 16 again yield a counterexample. Finally, (74) holds whenever the conditions of Corollary 13 are satisfied.

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